

# A Model for Stars of Interacting Bosons and Fermions

Cláudio M. G. de Sousa <sup>1</sup> and J. L. Tomazelli <sup>2</sup>

International Centre for Theoretical Physics, P.O. Box 586, 34100  
Trieste, Italy

## Abstract

In this paper we introduce a current-current type interaction term in the Lagrangian density of gravity coupled to complex scalar fields, in the presence of a degenerated Fermi gas. For low transferred momenta such a term, which might account for the interaction among boson and fermion constituents of compact stellar objects, is subsequently reduced to a quadratic one in the scalar sector. This procedure enforces the use of a complex radial field counterpart in the equations of motion. The real and the imaginary components of the scalar field exhibit different behaviour as the interaction increases. The results also suggest that the Bose-Fermi system undergoes a BCS-like phase transition for a suitable choice of the coupling constant.

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<sup>1</sup>On leave from Departamento de Física, Universidade de Brasília, 70910-900, Brasília-DF, Brazil *and* International Centre of Condensed Matter Physics, Universidade de Brasília, Caixa Postal 04667, 70919-970, Brasília – DF, Brazil (e-mail: [claudio@iccmp.br](mailto:claudio@iccmp.br))

<sup>2</sup>On leave from Instituto de Física Teórica–UNESP, Rua Pamplona 145, 01405–900, São Paulo–SP, Brazil (e-mail: [tomazell@axp.ift.unesp.br](mailto:tomazell@axp.ift.unesp.br))

# 1. Introduction

In modern approaches to astrophysics and cosmology, concepts that were initially restricted to elementary particle physics acquire a wider meaning, particularly in inflationary models. These models give predictions for the mass density of the present Universe larger than that observed, if one presumes that it is close to the critical value. This suggests that there is a large amount of hidden matter, which has not been detected so far.

Among the possible candidates for the so called dark matter are boson stars<sup>[1]</sup>, which consist of gravitational bound states of scalar particles. Possibly, such structures were formed through gravitational collapse in the early Universe and may appear also in the core of composite objects, whose external envelope are made of standard baryonic matter.

Since in addition to the bosons there were also fermions in the primordial gas, we would expect boson-fermion stars to prevail. This system was studied in detail by Henriques *et al* [2]. In their work it has been shown that the properties of boson-fermion stars are qualitatively the same, irrespective of the addition of a self-coupling term for bosons. Nevertheless, it seems that there is no modeling for explicitly dealing with the interaction between bosons and fermions in such systems.

In this work we introduce an effective coupling between bosons and fermions to afford a more realistic description of the system and compare our results with the ones in the current literature. In section 2 we construct the energy-momentum tensor for interacting fermions and bosons using the Schwarzschild metric. In section 3 we obtain the evolution equations for the coefficients of the metric and for the fields. In section 4 we exhibit the results of numerical simulations for the corresponding dynamical system. Finally, in section 5 we discuss our results and compare with those obtained without taking the interaction into account.

## 2. The Boson-fermion Interaction

Before introducing the interaction term, we outline the non-interactive boson-fermion model. We assume the metric to be the standard Schwarzschild one,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 . \quad (1)$$

Our sign and conventions are the same as those used by Liddle<sup>[3]</sup>. The Lagrangian for the complex scalar field with no explicit interaction is given by

$$\mathcal{L} = \frac{R}{16\pi G} - \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi , \quad (2)$$

where

$$\Phi(r, \tau) = \phi(r) e^{-i\omega\tau} \quad (3)$$

and for our purposes  $\phi$  is a complex scalar field.

For the combined bosons-fermions star we consider that the fermions are described by a perfect fluid with energy density  $\rho$  and pressure  $p$  as proposed by Chandrasekhar<sup>[4]</sup>:

$$\rho = K(\sinh t - t) , \quad (4)$$

$$p = \frac{K}{3}(\sinh t - 8 \sinh \frac{t}{2} + 3t) ,$$

where  $t$  is a parameter,  $K = m_n^4/32\pi^2$ , and  $m_n$  is the fermion mass (to be considered as that of neutrons for illustrative purposes). The evolution equation for the fermions is given by an equation of state, namely we have

$$p' = -\frac{1}{2}(\rho + p)\frac{B'}{B} , \quad (5)$$

where the primes stand for derivatives with respect to  $r$ .

The corresponding energy-momentum tensor for the bosons and fermions without interaction reads as

$$T_{\mu\nu}^{(0)} = T_{\mu\nu}^B + T_{\mu\nu}^F , \quad (6)$$

with

$$T_{\mu\nu}^B = \partial_\nu \Phi^* \partial_\mu \Phi + \partial_\mu \Phi^* \partial_\nu \Phi - g_{\mu\nu}(\partial_\lambda \Phi^* \partial^\lambda \Phi + m^2 \Phi^* \Phi) , \quad (7)$$

$$T_{\mu\nu}^F = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} , \quad (8)$$

where superscripts  $B$  and  $F$  label bosons and fermions from now on.

At this point we introduce the following interaction term in the Lagrangian density

$$\mathcal{L}^{int} = \lambda J_\mu(\Phi) j^\mu(\psi) , \quad (9)$$

where

$$J_\mu(\Phi) = i(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) , \quad (10)$$

$$j^\mu(\psi) = \bar{\psi} \gamma^\mu \psi \quad (11)$$

which represent the boson and fermion currents, respectively, while the  $\gamma$ 's are the usual Dirac matrices, which satisfy  $\gamma^{0\dagger} = -\gamma^0$  and  $\gamma^{i\dagger} = \gamma^i$  ( $i = 1, 2, 3$ ). This is a typical contact or current-current interaction between bosons and fermions where the coupling constant has dimension  $[\lambda] = M^{-2}$ . We emphasize that in dealing with a nonrenormalizable interaction term an energy scale must be introduced when we consider quantum corrections; we find a similar situation in the pure Einstein-Hilbert gravity. Note that this point is of no relevance since we are giving a semiclassical treatment to the problem.

The complete Lagrangian density, including the interaction term, is invariant under global  $U(1)$  gauge transformations. For low transferred momenta we may replace Eq.(8) by

$$j^\mu(\psi) = \bar{\psi} \Gamma^\mu \psi , \quad (12)$$

where

$$\Gamma^0 \equiv iu^0 , \quad \Gamma^i \equiv u^i . \quad (13)$$

The four-vector  $u^\mu = (u^0, u_r, u_\theta, u_\varphi)$  is the four-velocity of the fermion fluid.

The contribution of the interaction term for the energy-momentum tensor is given by

$$T_{\mu\nu}^{int} = -i\lambda(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) \bar{\psi} \Gamma_\nu \psi + g_{\mu\nu} J_\lambda j^\lambda , \quad (14)$$

which together with (6) give the total energy-momentum tensor

$$T_{\mu\nu} = T_{\mu\nu}^0 + T_{\mu\nu}^{int} .$$

Using (3) and its complex conjugate in (14) we arrive at

$$T_0^{0\ int} = i \frac{\lambda}{\sqrt{A}} (\phi^* \phi' - \phi \phi'^*) \bar{\psi} \psi , \quad (15)$$

$$T_1^{1\ int} = -2\omega \frac{\lambda}{\sqrt{B}} \phi^* \phi \bar{\psi} \psi . \quad (16)$$

Instead of the bilinear operator  $\bar{\psi} \psi$  we take the ground-state of the fermions system  $\langle \bar{\psi} \psi \rangle_F$  in the semi-classical approximation. In this limit we replace this quantity by the average fermion density  $\bar{n}_F$ .

Next, we write the complex scalar fields as

$$\phi = \phi_1 + i\phi_2 , \quad \phi^* = \phi_1 - i\phi_2 ,$$

which yields

$$T_0^{0\ int} = \frac{\alpha}{\sqrt{A}} (\phi_2 \phi_1' - \phi_1 \phi_2') , \quad (17)$$

$$T_1^{1\ int} = -\frac{\omega \alpha}{\sqrt{B}} (\phi_1^2 + \phi_2^2) , \quad (18)$$

where  $\alpha \equiv 2\bar{n}_F \lambda$ . The remaining nonvanishing contributions for the total energy momentum-tensor follows immediately from (6).

### 3. Evolution Equations

Now we are ready to write the evolution equations for our system. With an appropriate redefinition of the dynamical variables and parameters, namely

$$\begin{aligned} x &= mr , \\ \sigma(x) &= \sqrt{8\pi G} \phi(r) , \\ \bar{\rho}(t) &= \frac{4\pi G}{m^2} \rho(t) , \\ \bar{p}(t) &= \frac{4\pi G}{m^2} p(t) , \\ \bar{\alpha} &= \frac{\alpha}{m} , \quad w = \frac{\omega}{m} , \end{aligned}$$

the equations read

$$A' = xA^2 \left[ 2\bar{\rho} + \left( \frac{w^2}{B} + 1 \right) \sigma^2 + \frac{s^2}{A} - \frac{\bar{\alpha}}{\sqrt{A}} (\sigma_2 \sigma_1' - \sigma_1 \sigma_2') \right] - \frac{A}{x} (A - 1) , \quad (19)$$

$$B' = xAB \left[ 2\bar{p} + \left( \frac{w^2}{B} - 1 \right) \sigma^2 + \frac{s^2}{A} - \frac{w\bar{\alpha}}{\sqrt{B}} \sigma^2 \right] + \frac{B}{x} (A - 1) , \quad (20)$$

$$\sigma_1' = s_1 , \quad (21)$$

$$\sigma_2' = s_2 , \quad (22)$$

$$s_1' = -A \left( \frac{w^2}{B} - 1 + \frac{\bar{\alpha}w}{\sqrt{B}} \right) \sigma_1 + \bar{\alpha}\sqrt{A}s_2 + \left[ \frac{1}{2} \left( \frac{A'}{A} - \frac{B'}{B} \right) - \frac{2}{x} \right] s_1 + \frac{\bar{\alpha}\sqrt{A}}{2} \left( \frac{B'}{2B} + \frac{2}{x} \right) \sigma_2 , \quad (23)$$

$$s_2' = -A \left( \frac{w^2}{B} - 1 + \frac{\bar{\alpha}w}{\sqrt{B}} \right) \sigma_2 - \bar{\alpha}\sqrt{A}s_1 + \left[ \frac{1}{2} \left( \frac{A'}{A} - \frac{B'}{B} \right) - \frac{2}{x} \right] s_2 - \frac{\bar{\alpha}\sqrt{A}}{2} \left( \frac{B'}{2B} + \frac{2}{x} \right) \sigma_1 , \quad (24)$$

$$t' = -2\frac{B'}{B} \frac{\sinh t - 2 \sinh \frac{t}{2}}{\cosh t - 4 \cosh \frac{t}{2} + 3} , \quad (25)$$

where

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 , \quad s^2 = s_1^2 + s_2^2 .$$

These equations form a non-autonomous system of non-linear first-order differential equations, which cannot be linearized due to the quadratic terms involved. Equations (19) and (20) are the Einstein equations; equations (21)–(24) correspond to the Klein-Gordon equation, whereas (25) gives the evolution of the fermion energy density and pressure. In the above set of equations the dynamical variables and parameters are dimensionless. From now on the primes stand for derivatives with respect to  $x$ .

It is noteworthy that these equations are invariant under the scale transformation

$$B \longrightarrow \eta B , \quad w \longrightarrow \sqrt{\eta} w ,$$

even after the inclusion of the interaction term. Since the initial value  $B_0$  is undetermined, this permits its redefinition during numerical calculations, in such a way to obtain the asymptotic values for the metric coefficients converging to those of a flat space.

## 4. Numerical Results

In this section we present the numerical simulation results of the above set of equations. The set of equations were solved by using the fifth-order Runge-Kutta method; we also use an appropriate “shooting method” to infer the value of  $w$ , according to an initial value of  $B_0$ .

For further details on the numerical criteria, the interested reader is referred to Ruffini and Bonazzola<sup>[5]</sup>. We require that the metric to be asymptotically flat and that the scalar fields as well as their derivatives to vanish at infinity.

Close to the singularity at the origin, which is inherent to Schwarzschild metric, we see that the last terms in (19) and (20) are dominant. In this region  $A \sim x/(x - \text{const.})$  and  $B \sim (x - \text{const.})/x$ , so that  $B$  and  $B'$  diverges and  $A$  becomes oscillating. Hence, for numerical purposes it is convenient to start with  $x$  slightly shifted from the origin.

In Fig.1 we show the results first obtained in Ref.[5] for a typical boson star, where we note that there is no overlap between  $\sigma$  and  $\sigma'$ , which characterizes the ground state of the system. Throughout the simulations the initial values are:  $\sigma_1^0 = \sigma_2^0 = 0.32$ ,  $s_1^0 = s_2^0 = 0$  and  $A_0 = 1$ .

In Fig.2 we show the ground-state of a boson-fermion star for  $t_0 = 7$ . In this case  $B_0 = 0.036$  and  $w = 1.098$ .

Fig.3 displays the same curves for  $t_0 = 8.7$  which corresponds to  $B_0 = 0.01$  and  $w = 1.327$ . Notice that the peak in the curve for  $A$  has increased and has been shifted towards the origin, while  $\sigma$  and  $\sigma'$  approach to zero faster. Larger values of  $t_0$  mean higher fermion energy densities, so that the fermion contribution to the energy-momentum tensor becomes dominant. In this sense we would expect curves like those given in Fig.4 to be in agreement with the pioneering results of Oppenheimer and Volkoff for neutron stars.

Figures 5 and 6 exhibit the results after introducing the interaction term, for the choice  $t_0 = 8.7$ , and different values of the dimensionless coupling constant  $\bar{\alpha}$ . When we switch the interaction on at small values of  $\bar{\alpha}$ ,  $w$  increases, as shown in table 1. On the other hand  $t$  vanishes at a smaller  $x$ .

If we continue to increase the interaction we observe that, for a certain value of  $\bar{\alpha}$ ,  $w$  suddenly decreases, suggesting that there is a critical value of  $\bar{\alpha}$ , in the range  $10^{-3} - 10^{-2}$ , in which the system experiences a second-order phase transition, corresponding to boson-fermion pair formation. This phenomenon is similar to that observed in the BCS theory; this is not so surprising since we have started with a quartic interaction term of the form (9). In Fig.6 (b) and (c) we can observe the splitting of the real and imaginary parts of the scalar field that also occurs in such interval. For completeness, we also exhibit the phase space diagrams for the metric coefficients and for the scalar fields at  $\bar{\alpha} = 10^{-2}$  in Fig.7.

$\bar{\alpha}$	$w$	$B_0$
$10^{-15}$	1.482	0.11
$10^{-9}$	1.557	0.11
$10^{-6}$	1.565	0.011
$10^{-3}$	1.335	0.010
$10^{-2}$	1.060	0.009

**Table 1:** values of  $w$  and  $B_0$  for different coupling constants  $\bar{\alpha}$  and  $t_0 = 8.7$ .

## 5. Concluding Remarks

In this work we studied a model for boson-fermion stars in interaction, in the region of low frequencies, and observed the behaviour of the system for increasing values of the coupling constant  $\bar{\alpha}$ . At small values of  $\bar{\alpha}$  there is no significant changes in the system; as  $\bar{\alpha}$  becomes larger and larger the ground state energy of the system increases and the fermion energy density and pressure vanish at smaller distances. As a result, the fermion star is confined to a smaller region, after which the scalar fields are still present.

When  $\bar{\alpha}$  reaches a critical value, the ground state energy of the system suddenly decreases, indicating the possible occurrence of a second-order phase transition. In this case, we might have supplied enough energy to bind bosons and fermions together.

From the results outlined above it is possible to compute the mass and the radius for typical boson-fermions stars; this is the subject of a forthcoming paper. Another interesting matter, is to develop a qualitative approach to our evolution equations with Schwarzschild metric, in order to compare the phase diagrams with our corresponding numerical results.

A more general treatment of boson-fermion gravitationally bounded systems should incorporate spin effects by considering the full Dirac Lagrangian. However, it would be necessary to extend the system of differential equations from seven to fifteen, which might demand a great effort.

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## References

1. Ph. Jetzer, *Phys. Rep.* **220**, 4 (1992) 163;
2. A. B. Henriques, A. R. Liddle and R. G. Moorhouse, *Phys. Lett.* **B 233** (1989) 99;
3. A. R. Liddle and M. S. Madsen, *Int. J. Mod. Phys.* **D 1**, 1 (1992) 101;
4. S. Chandrasekhar, *Ap. J.* **140**, 2 (1964) 417;
5. R. Ruffini and S. Bonazzola, *Phys. Rev.* **187** (1969) 1767;
6. J. R. Oppenheimer and G. M. Volkoff, *Phys. Rev.* **55** (1939) 374.

## Figures Captions

**Figure 1:** The values of the fields and metric coefficients in a typical boson star configuration. The corresponding value of  $w$  is around 0.826.

**Figure 2:** Fields and metric coefficients for a typical boson-fermion star, with  $t_0 = 7$ . In this case, the value of  $w$  has increased to 1.098.

**Figure 3:** The same curves as those of Fig.2, for  $t_0 = 8.7$ .

**Figure 4:** The evolution of  $t$  for different initial values.

**Figure 5:** Fields and metric coefficients for a boson-fermion star when we switch on the interaction at  $\bar{\alpha} = 10^{-9}$ .

**Figure 6:** The same curves as those shown in the last figure, for  $\bar{\alpha} = 10^{-2}$ .

**Figure 7:** Phase space diagrams (a) for the metric coefficients and (b) for the scalar fields.